MTH 203 - Quiz 4 Solutions

1. Let $R_{x,\theta} \in \text{Sym}(\mathbb{R}^2)$ denote the counterclockwise rotation about a point $x \in \mathbb{R}^2$ by an angle θ.

   (a) Show that $R_{x,\theta}$ is a product of two reflections in $\text{Sym}(\mathbb{R}^2)$.

   (b) Show that for rotations $R_{x_1,\theta_1}, R_{x_2,\theta_2} \in \text{Sym}(\mathbb{R}^2)$ such that $\theta_1 + \theta_2 \neq 2k\pi$ for some $k \in \mathbb{Z}$, the product $R_{x_1,\theta_1} R_{x_2,\theta_2} \in \text{Sym}(\mathbb{R}^2)$ is also a rotation. [4+6]

   Solution. We provide a geometric proof to (a) and (b). The formal algebraic proof is left as an exercise.

   (a) It can be easily seen that the rotation $R_{x,\theta}$ is the product of reflections about the two lines ℓ_1 and ℓ_2 intersecting at x at an angle $\theta/2$, as shown in Figure 1 below.

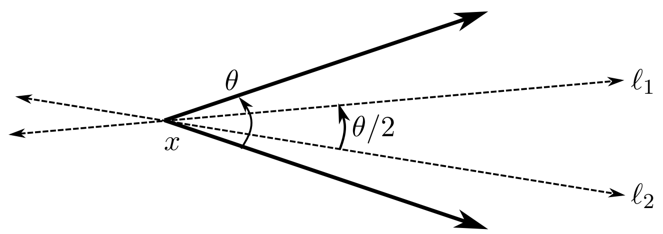

Figure 1: The rotation $R_{x,\theta}$ has a product of reflections.

   (b) For $1 \leq i \leq 3$, let $\mathcal{R}_i$ denote the reflection about the line ℓ_i shown in Figure 2 below.

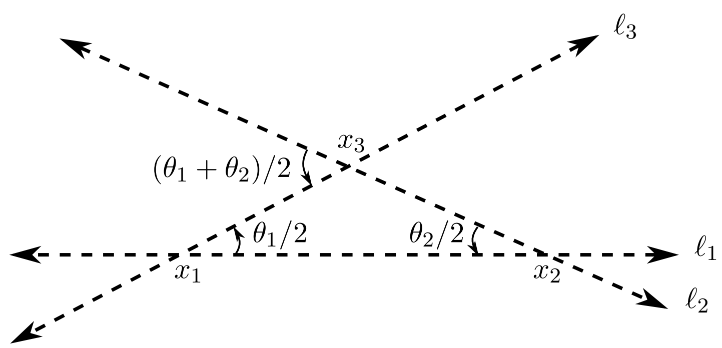

Figure 2: The product of the two rotations R_{x_1,θ_1} and R_{x_2,θ_2}.

From (a), we have $R_{x_1,\theta_1} = \mathcal{R}_3\mathcal{R}_1$ and $R_{x_2,\theta_2} = \mathcal{R}_1\mathcal{R}_2$. Thus, it follows that

$$R_{x_2,\theta_2}R_{x_1,\theta_1} = \mathcal{R}_3\mathcal{R}_2 = R_{x_3,\theta_1+\theta_2},$$

assuming that $\theta_1 + \theta_2 \neq 2k\pi$. (Verify that when $\theta_1 + \theta_2 = 2k\pi$, $R_{x_2,\theta_2}R_{x_1,\theta_1}$ is a translation.)

2. Let $R_{v,\theta} \in \text{Sym}(\mathbb{R}^3)$ denote the counterclockwise rotation about a vector $v \in \mathbb{R}^3$ by an angle θ.

   (a) For $i = 1, 2$, let $R_i = R_{v_i,\theta_i}$. Then show that $R_2R_1R_2^{-1}$ is a rotation about a vector in $\mathbb{R}^3$.

   (b) Show that a nontrivial normal subgroup of $\text{SO}(3, \mathbb{R})$ is infinite. [4+6]

   Solution. (a) Since R_1 fixes the vector v_1, we have:

   $$R_2R_1R_2^{-1}(R_2(v_1)) = R_2(R_1(v_1)) = R_2(v_1),$$

   from which it follows that $R_2R_1R_2^{-1}$ fixes the vector $R_2(v_1)$. Since $R_i \in \text{SO}(3, \mathbb{R})$, it follows that $R_2R_1R_2^{-1} \in \text{SO}(3, \mathbb{R})$ has the eigenvector $R_2(v_1)$ corresponding to the eigenvalue $+1$. Thus, we know from the proof of 6.2 (ix) that $R_2R_1R_2^{-1}$ must be a rotation about $R_2(v_1)$ by an angle $\theta_2 + \theta_1 - \theta_2 = \theta_1$, that is, $R_2R_1R_2^{-1} = R_{R_2(v_1),\theta_1}$.

   (b) Let $H = \{M_1, \dots, M_n\}$ be a normal subgroup of $\text{SO}(3, \mathbb{R})$. Since each matrix M_i has 1 as eigenvalue, let its corresponding eigenvector be v_i. Thus, M_i represents a rotation in $\mathbb{R}^3$ about v_i by θ_i, i.e, $M_i = R_{v_i,\theta_i}$. Consider an $M \in \text{SO}(3, \mathbb{R})$ such that $M(v_j) \notin \{v_1, \dots, v_n\}$ for some j. (Why is such a choice possible?) Then by (a), $MM_jM^{-1} = R_{M(v_j),\theta_j} \notin H$, which contradicts the normality of H.